Nonlinear Partial Differential Equations 2023

On the occasion of J. M. Mazón 70th birthday

Abstracts

David Arcoya (Universidad de Granada)

Hardy potential versus lower order terms

Abstract: We summary the joint results with **Lucio Boccardo** and **Luigi Orsina** in [1] to study the regularizing effect in Dirichlet problems of some lower order terms satisfying the so-called Q-condition despite the presence of Hardy potentials in the right hand side.

References

[1] D. Arcoya, L. Boccardo, L. Orsina: Hardy potential versus lower order terms in Dirichlet problems: regularizing effects, Mathematics in Engineering, 5(1): 1–14.
https://doi.org/10.3934/mine.2023004

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Luccio Boccardo (Sapienza-Università di Roma)

TITLE TBA

ABSTRACT:

1

♠ *Dirichlet* - Let Ω be a bounded open set in $I\!R^N$, $N \geq 2$.

$$\begin{cases} -\mathrm{div}\big(M(x)\nabla u\big) = f(x), & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega; \end{cases} \tag{1.1}$$

♠ If $M(x)$ is smooth, the Calderon-Zygmund (see also recent results by Haim Brezis) theory states $\boxed{f \in L^m(\Omega) \Rightarrow u \in W_0^{1,m^*}(\Omega),\ 1 < m < \infty}$;

♠ If $M(x)$ is only bounded and elliptic, [B-Gallouet] proved the same result for $1 < m < \frac{2N}{N+2}$, that is for infinite energy solutions.

♠ ! The case $m > \frac{2N}{N+2}$.

♡ *Nonlinear problems, Leray-Lions operators $\mathcal{A}$ in $W_0^{1,p}(\Omega)$, $\mathbf{1}< p < N$* - The simplest example of $\mathcal{A}$ is the p–Laplace operator. Consider the b.v.p. for $\mathcal{A}$:

$$\begin{cases} -\mathrm{div}\big(a(x,\nabla u)\big) = f(x), & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega; \end{cases} \tag{1.2}$$

THEOREM 1.1. *[BG] ♢ Let $2 - \frac{1}{N} < p < N$. If*

$$\int_\Omega |f| \log(1 + |f|) < \infty, \tag{1.3}$$

then there exists a distributional solution $u \in W_0^{1,\frac{N(p-1)}{N-1}}(\Omega)$ of (1.2).

THEOREM 1.2 (Calderon-Zygmund theory for infinite energy sol. (BG)). *♢ If $f \in L^m(\Omega)$, $\sup\left(1, \frac{N}{N(p-1)+1}\right) < m < \frac{Np}{pN+p-N} = (p^\star)'$, $p > 1 + \frac{1}{m} - \frac{1}{N}$, then there exists a distributional solution $u \in W_0^{1,(p-1)m^*}(\Omega)$ of (1.2).*

THEOREM 1.3. *♣[BG] Let $f \in L^m(\Omega)$, $m = \frac{N}{N(p-1)+1}$, $1 < p < 2 - \frac{1}{N}$. Then there exists a distributional solution $u \in \mathbf{W_0^{1,1}(\Omega)}$ of (1.2).*

THEOREM 1.4. *♣[BG] Assume (1.3) and $p = 2 - \frac{1}{N}$. Then there exists a distributional solution $u \in \mathbf{W_0^{1,1}(\Omega)}$ of (1.2).*

REMARK 1.5. $\lim_{p\to 1} \frac{N}{N(p-1)+1} = N$, $\quad \lim_{p\to 2-\frac{1}{N}} \frac{N}{N(p-1)+1} = 1$

♣ *Neumann* -(B-Moreno:dedicated to 60-Mazon/joint work+Mazon)

2. RECENT RESULTS ♡

Verena Bögelein (Universität Salzburg)

Gradient regularity for widely degenerate elliptic and parabolic systems

Abstract:

In this talk we consider widely degenerate elliptic systems of the form

$$\operatorname{div}\left((|Du|-1)_+^{p-1} \frac{Du}{|Du|} \right) = f, \qquad p > 1$$

and its parabolic couter-part. Such PDEs are motivated for instance by congested traffic dynamics problems or models of gas filtration with nonlinear effects. Continuity properties of the gradient have been investigated in the scalar elliptic setting by Santambrogio & Vespri and Colombo & Figalli. In this talk we establish gradient regularity in the vector valued case for the elliptic as well as for the parabolic problem. More precisely, we can show that $\mathcal{K}(Du)$ is continuous for any continuous function $\mathcal{K}\colon \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ vanishing on the set $\{\xi \in \mathbb{R}^{Nn} : |\xi| \leq 1\}$. This is joint work with F. Duzaar, R. Giova and A. Passarelli di Napoli.

Leon Bungert (Universität Würzburg)

The nonlocal geometry of adversarial machine learning

Abstract: It is well-known that, despite their aptness for complicated tasks like image classification, modern neural networks are prone to insusceptible input perturbations (a.k.a. adversarial attacks) which can lead to severe misclassifications. Adversarial training is a method to obtain classifiers which are robust against such attacks. In this talk I will show that in the binary classification setting the method can be rephrased as geometric regularization problem, involving a nonlocal perimeter of Minkowski type. I will present existence and regularity theorems for minimizers, study local asymptotics of the nonlocal perimeter using Gamma-convergence, and discuss probabilistic relaxations which correspond to more classical notions of nonlocal perimeters.

Xavier Cabré (ICREA and Universitat Politècnica de Catalunya)

Stable solutions to semilinear elliptic equations: quantitative proofs of regularity up to dimension 9

Abstract: The regularity of stable solutions to semilinear elliptic PDEs has been studied since the 1970's. It was initiated by a work of Crandall and Rabinowitz, motivated by the Gelfand problem in combustion theory. The theory experienced a revival in the mid-nineties after new progress made by Brezis and collaborators. I will present these developments and my work in collaboration with Figalli, Ros-Oton, and Serra, which finally establishes the regularity of stable solutions up to the optimal dimension 9. I will also describe a more recent paper of mine which provides full quantitative proofs of the regularity results. I will finally comment on similar progress and open problems for related equations.

José Carmona Tapia (Universidad de Almería)

ELLIPTIC BVP'S WITH A CONTINUUM OF BIFURCATION POINTS LACKING EMANATING CONTINUA

ABSTRACT:

We study the global behavior of continua of solutions to the semilinear elliptic problem

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded, open subset of $\mathbb{R}^N$ $(N \geq 3)$ and $f : (0, +\infty) \to [0, +\infty)$ is a continuous real function having with a countable infinite set of positive zeroes. For the model cases $f(t) = t^r(1 + \sin t)$ and $f(t) = t^r\left(1 + \sin \frac{1}{t}\right)$ with $r \geq 0$ we show that there are some values of r for which every $\lambda > 0$ is a local bifurcation point (either from infinity or from zero) but there is no global bifurcation point.

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[3] J. García-Melián, L. Iturriaga, *Multiplicity of solutions for some semilinear problems involving nonlinearities with zeros*, Israel J. Math. 210 (2015), no. 1, 233–244.
[4] P. Hess, *On multiple positive solutions of nonlinear elliptic eigenvalue problems*, Comm. Partial Differential Equations 6 (1981), no. 8, 951–961.

Frank Duzaar (Universität Erlangen-Nürnberg)

GRADIENT ESTIMATES FOR A CLASS OF DOUBLY NONLINEAR PARABOLIC EQUATIONS

ABSTRACT: In this talk, we address the local behavior of nonnegative weak solutions of the doubly nonlinear parabolic equation

$$\partial_t u^q - \operatorname{div}\left(|Du|^{p-2}Du\right) = 0.$$

In particular, we will discuss Hölder estimates for the gradient of weak solutions in the supercritical fast diffusion regime $0 < p - 1 < q < \frac{N(p-1)}{(N-p)_+}$, where N is the space dimension. Two main components of these regularity estimates are a time-insensitive Harnack inequality specific to this regime and Schauder estimates for the parabolic p-Laplace equation.

These results are the outcome of a collaboration with Verena Bögelein (Salzburg), Ugo Gianazza (Pavia), Naian Liao (Salzburg) and Christoph Scheven (Duisburg-Essen).

Abderrahim Elmoatazi (Université de Caen-Normandie)

PDEs based Image processing for cultural Heritage applications and accessibilty

Abstract: This talks showcases PDEs based image processing algorithms, in the light of applications and accessibility to cultural heritage artifacts. We emphasize the potential PDEs on graphs to solve many problems such as interpolation, segmentation and of multimodal 2D image registration and fine-scale 3D-reconstruction techniques, with the aim to ease the work of historians and museum curators, as well as to make artifacts more accessible to the general public or to visually impaired people. we then focuses on the Bayeux Tapestry, a world-famous medieval wool embroidery included in UNESCO's Memory of the World register, and of fundamental importance for both the scientific community and the general public.

Raul Ferreira (Universidad Complutense de Madrid)

Fujita exponent for a mixed local and nonlocal semilinear heat equation

Abstract: We present some aspect about the blow-up phenomena of the problem

$$\begin{cases} u_t = \mathcal{L}u + u^p & x \in \mathbb{R}^d, t > 0 \\ u(x,0) = g(x) \end{cases}$$

where the diffusion operator $\mathcal{L} = a\Delta - b(-\Delta)^s$.

Joint works with L. M. del Pezzo and F. del Teso.

Lorenzo Giacomelli (Sapienza-Università di Roma)

Droplet models with singular potentials: equilibria and travelling waves

Abstract: We look at spreading phenomena under the action of singular potentials modeling repulsion between the liquid/gas interface and the substrate. First we briefly review the statics: depending on the form of the potential, the macroscopic profile of minimizers (when they exist) can be either droplet-like or pancake-like, with a transition profile between the two at zero spreading coefficient. Then we focus on the dynamics, assuming null slippage at the contact line. Based on formal arguments and numerical evidences, we report that travelling-wave solutions generically exist and have finite rate of bulk dissipation, indicating that singular potentials stand as an alternative solution to the contact-line paradox. In agreement with steady states, travelling-waves havefinite energy for mild singularities. Time permitting, we also discuss a selection criterion for travelling waves, based on thermodynamically consistent contact-line conditions modeling friction at the contact line.

Based on joint works with Riccardo Durastanti.

Wojciech Górny (Universtät Wien)

Weak solutions to gradient flows in metric measure spaces

Abstract: Due to the fact that in a metric space there is (in general) no notion of directional derivatives, the definition of solutions to gradient flows in metric measure spaces necessarily avoid their direct use. The most classical example is the heat flow, or more generally the p-Laplacian evolution equation, which has been studied as the gradient flow in L^2 of the p-Cheeger energy. Typically, solutions are defined using the semigroup theory through a subdifferential of the energy. Other popular approaches include variational solutions and the weighted energy-dissipation formalism. In this talk, using the first-order differential structure on a metric measure space introduced by Gigli, we characterize the subdifferential in L^2 of the p-Cheeger energy. This gives rise to a new notion of solutions to the p-Laplacian evolution equation in metric measure spaces. In the case p=1, we introduce a metric analogue of the Anzellotti pairing and obtain a Green-Gauss formula, which is then used in place of Gigli's structure to characterise the 1-Laplacian operator and study the total variation flow. The talk is based on a series of papers written jointly with J. M. Mazón.

Daniel Hauer (The University of Sydney)

An extension problem for the logarithmic Laplacian

Abstract: Motivated by the fact that for positive s tending to zero the fractional Laplacian converges to the identity and for s tending to 1 to the local Laplacian, Chen and Weth [Comm. PDE 44 (11), 2019] introduced the logarithmic Laplacian as the first variation of the fractional Laplacian at s=0. In particular, they showed that the logarithmic Laplacian admits an integral representation and can, alternatively, be defined via the Fourier-transform with a logarithmic symbol. The logarithmic Laplacian turned out to be an important tool in various mathematical problems; for instance, to determine the asymptotic behavior as the order s tends to zero of the eigenvalues of the fractional Laplacian equipped with Dirichlet boundary conditions (see, e.g., [Feulefack, Jarohs, Weth, J. Fourier Anal. Appl. 28(2), no. 18, 2022]), in the study of the logarithmic Sobolev inequality on the unit sphere [Frank, König, Tang, Adv. Math. 375, 2020], or in the geometric context of the 0-fractional perimeter, see [De Luca, Novaga, Ponsiglione, ANN SCUOLA NORM-SCI 22(4), 2021].

Caffarelli and Silvestre [Comm. Part. Diff. Eq. 32(7-9), (2007)] showed that for every sufficiently regular u, the values of the fractional Laplacian at u can be obtained by the co-normal derivative of an s-harmonic function w_u on the half-space (by adding one more space dimension) with Dirichlet boundary data u. This extension problem represents the important link between an integro-differential operator (the nonlocal fractional Laplacian) and a local 2nd-order differential operator. This property has been used frequently in the past in many problems governed by the fractional Laplacian.

In this talk, I will present an extension problem for the logarithmic Laplacian, which shows that this nonlocal integro-differential operator can be linked with a local Poisson problem on the (upper) half-space, or alternatively (after reflection) in a space of one more dimension. As an application of this extension property, I show that the logarithmic Laplacian admits a unique continuous property.

The results presented here were obtained in joint work with Huyuan Chen (Jiangxi Normal University, China) and Tobias Weth (Goethe-Universität Frankfurt, Germany).

Noureddine Igbida (Université de Limoges)

Steepest descent through Wasserstein distance vs minimum flow, and applications in PDE

Abstract: The aim of this talk is to discuss how the Wasserstein distance and the minimum flow problem can be used to model physical phenomena using PDEs with the steepest descent algorithm. While these two approaches produce the same models in some standard cases, we will show that this is not always true. In this talk, we focus on some applications in crowd motion and congestion.

Delio Mugnolo (FernUniversität in Hagen)

p-Laplacians on graphs and networks

Abstract: I will provide an introductory overview of homogeneous operators in the context of combinatorial and metric graphs. In several relevant instances, these operators give rise to contractive nonlinear semigroups. In this presentation, I will cover established properties of the associated parabolic problems, and delve into more subtler aspects concerning the spectrum of these operators and the geometry of their eigenfunctions.

Francescantonio Oliva (Sapienza-Università di Roma)

First order terms in 1-Laplace problems involving general nonlinearities

Abstract: We discuss existence of finite energy solutions to 1-Laplace Dirichlet problems involving general nonlinearities which are possibly singular. In particular, we are interested into the regularizing effects given to the solutions by zero and first order terms, and their interplay.

Francesco Petitta (Sapienza-Università di Roma)

Unexpected phenomena in a one dimensional singular elliptic equation

Abstract: We aim to show existence and non-existence results for one-dimensional problems as

$$\begin{cases} -(a(x)u_x)_x = -\phi(u)_x - g_x & \text{in } [0,L], \\ u(0) = u(L) = 0\,, \end{cases}$$

a is a positive bounded function, $g \in L^2(0,L)$, and $\phi : \mathbb{R} \mapsto \mathbb{R} \cup \{+\infty\}$ which is continuous function outside $s = 0$. Also, some relevant and quite unexpected qualitative and quantitative facts concerning such problems and their solutions are discussed, as well as some relevant open problems. The results presented are part of an ongoing project in collaboration with Daniela Giachetti, Pedro J. Martínez-Aparicio, and François Murat.

Fernando Quirós (Universidad Autónoma de Madrid)

IMPROVED ASYMPTOTIC ESTIMATES FOR THE HEAT EQUATION IN EXTERIOR DOMAINS

ABSTRACT: Giving good lower bounds for the Dirichlet heat kernel in the complement of a compact set is surprisingly a relatively new result. We use entropy methods and some recent advances in logarithmic Sobolev inequalities to improve the available results, obtaining optimal asymptotic bounds for large times with an explicit approach rate. This is a work in collaboration with José A. Cañizo and Alejandro Gárriz.

José C. Sabina de Lis (Universidad de La Laguna)

ASYMPTOTIC BEHAVIOR OF THE HIGHER EIGENVALUES TO Δ_p AS $p \to 1$.

ABSTRACT:

The limit behavior of the Dirichlet eigenvalues to $-\Delta_p$ as $p \to 1$ has been recently studied in the literature ([1], [2]). Furthermore, such analysis has been extended to cover its natural 'nonlinear' perturbations ([4], [3]). In this talk we are reviewing new results on the dependence on p of the higher eigenvalues to other boundary value problems. Such an study also addresses the limit trend as $p \to 1$. This is a joint research with Sergio Segura de León.

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Juan Soler (Universidad de Granada)

Recent results about mean-field limits for multi-agent systems on graphs

Abstract: This talk deals with the derivation of the mean-field limit for multi-agent systems on a large class of sparse graphs. More specifically, the case of non-exchangeable multi-agent systems consisting of non-identical agents is addressed. It also examines the major challenge of considering singular interactions (as in the case of the Navier-Stokes or Keller-Segel equations). The analysis encompasses not only PDEs and stochastic analysis but also graph theory through a new concept of limits of sparse graphs (extended graphons) that reflect the structure of the connectivities in the network. This has a crucial impact on collective dynamics. The associated observables completely entangle the limit of connectivities with the initial conditions and solve an independent hierarchy of equations. They naturally extend the notion of marginals, and hierarchy of marginals, to non-exchangeable systems. Some of the main restrictive hypothesis in the previous literature on the connectivities between the agents (dense graphs) and the cooperation between them (symmetric interactions) are removed.

Marcos Solera Diana (Universitat de València)

Weak solutions of anisotropic inverse mean curvature flow as limits of p-capacitary potentials

Abstract: We construct weak solutions of the anisotropic inverse mean curvature flow (A-IMCF) under very mild assumptions both on the anisotropy (which is simply a norm in $\mathbb{R}^N$ with no ellipticity nor smoothness requirements, in order to include the crystalline case) and on the initial data. By means of an approximation procedure introduced by Moser, our solutions are limits of anisotropic p-harmonic functions or p-capacitary functions (after a change of variable), and we get uniqueness both for the approximating solutions (i.e., uniqueness of p-capacitary functions) and the limiting ones. Our notion of weak solution still recovers variational and geometric definitions similar to those introduced by Huisken-Ilmanen, but requires to work within the broader setting of BV -functions. Despite of this, we still reach classical results like the continuity and exponential growth of perimeter, as well as outward minimizing properties of the sublevel sets. Moreover, by assuming the extra regularity given by an interior rolling ball condition (where a sliding Wulff shape plays the role of a ball), the solutions are shown to be continuous and satisfy Harnack inequalities. Finally, examples of explicit solutions are built.

Jose Miguel Urbano (KAUST and Universidade de Coimbra)

Shaping the free boundary on singular elliptic problems

Abstract: We consider free boundary models arising from minimising energy functionals with varying singularities. For measurable and bounded singular exponents, we establish the local $C^{1,\alpha}$-regularity of local minima. Sharp regularity estimates along the free boundary are also obtained under a mild continuity assumption on the exponent. These estimates vary point-by-point along the interface, leading to multiple free boundary geometries.